How Changing Prudence and Risk Aversion Affect Optimal Saving

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Abstract

We show how optimal saving in a two-period model is affected when prudence and risk aversion of the underlying utility function change. Increasing prudence alone will induce higher savings only if, for certain combinations of the interest rate and the pure time discount rate, there is distributional neutrality between the two periods. Otherwise, changes of risk aversion that affect the distribution between the periods must also be taken into account.

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1 Introduction

It is well-known that a mean-preserving spread of risky exogenous future wealth yields higher savings if the investor’s von Neumann-Morgenstern utility function is prudent, i.e. if its third derivative is positive (see Leland, 1968, Sandmo, 1970, and Drèze and Modigliani, 1972).\footnote{See Eeckhoudt and Schlesinger (2008) on the necessity and sufficiency of prudence in the case of other types of risk changes.} Just like different utility functions may show different degrees of risk aversion, they may also show different degrees of prudence (see Kimball, 1990, and the exposition in Gollier, 2001). While a globally higher degree of prudence will lead to higher savings in some cases, this assertion is not generally true (see, e.g., Menegatti, 2001, 2007, and Hau, 2002).

In this paper we further explore, within the standard two period savings model with uncertain wealth in the later period, how the degree of prudence affects the optimal amount of savings. In particular, we show, that the relationship between the investor’s time discount rate and the interest rate plays a crucial role for the effects a change of a von Neumann-Morgenstern utility function has on savings.

Only if the time discount rate is approximately equal to the interest rate it is ensured that a higher degree of prudence induces higher savings. Otherwise, as consumption smoothing across time matters in these cases, changes of risk aversion of the von-Neumann-Morgenstern utility function have to be taken into account as well.

The plan of the paper is as follows. Section 2 describes the model and proves our main results. Section 3 discusses how changes in prudence and changes in risk aversion interact in the determination of savings. Section 4 provides a general qualification by considering a number of irrelevance results. Section 5 concludes.

2 Model and Results

Consider the standard optimal savings model under uncertainty with two periods. The certain wealth in period 0 is denoted \( w_0 \), the uncertain wealth in period 1 is \( \tilde{w}_1 \). The von Neumann utility function \( u(c) \) is defined on \( \mathbb{R}^+ \), three times differentiable, with \( u'(c) > 0 \) and \( u''(c) < 0 \) for all consumption levels \( c > 0 \). The constant marginal rate of transformation
between consumption in the two periods and the pure time discount factor are given by $\rho \equiv 1 + r$ and $\beta \equiv 1 / (1 + \delta)$, respectively, where $r$ is the interest rate and $\delta$ is the investor’s pure rate of time discount. With $s$ denoting the amount of savings, the objective function is

$$u(w_0 - s) + \beta Eu(\bar{w}_1 + \rho s).$$

(1)

Assume that maximizing (1) with respect to $s$ yields an interior solution $s_u^*$, which is then characterized by the first order condition

$$u'(w_0 - s_u^*) = \beta \rho E u'(\bar{w}_1 + \rho s_u^*).$$

(2)

We now analyze how optimal savings change if the utility function $u(c)$ is substituted by another utility function $v(c)$, which has the same properties as $u(c)$ but shows a globally higher degree of prudence, i.e.

$$-\frac{v'''(c)}{v''(c)} > -\frac{u'''(c)}{u''(c)}$$

(3)

holds for all consumption levels. The following Proposition states our main result.

**Proposition 1.** An increase in prudence generates higher savings if either i) $\beta \rho$ is sufficiently close to 1, ii) $\beta \rho < 1$ and risk aversion increases, or iii) $\beta \rho > 1$ and risk aversion decreases.

**Proof.** In preparation of the proof, let $\hat{w}_1$ be defined by

$$u'(\hat{w}_1) = E u'(\bar{w}_1 + \rho s_u^*).$$

(4)

This precautionary equivalent wealth level is the certainty equivalent of consumption $\bar{w}_1 + \rho s_u^*$, taking $-u'(c)$ as the utility function.\(^2\) We then have

$$\beta \rho < 1 \ (= 1, > 1) \Rightarrow \frac{\hat{w}_1}{w_0 - s_u^*} < 1 \ (= 1, > 1),$$

(5)

since (2) and (4) imply $u'(\hat{w}_1) = \frac{u'(w_0 - s_u^*)}{\beta \rho}$ and $u(c)$ is concave.\(^2\)

\(^2\)The wealth level $\hat{w}_1$ is related to the well-known “precautionary equivalent premium” $\psi$ via $\hat{w}_1 = E\bar{w}_1 + \rho s_u^* - \psi(\rho s_u^*, u, \bar{w}_1)$ (see Kimball, 1990, and Gollier, 2001, 128).
(i) As \( v(c) \) is more prudent than \( u(c) \), \(-v'(c)\) is more risk averse than \(-u'(c)\). Using a standard result on changes of risk aversion (see, e.g., Gollier, 2001, 21), we obtain, in the case of \( \beta \rho = 1 \),

\[
v'(w_0 - s_u^*) = v' (\hat{w}_1) < Ev' (\hat{w}_1 + \rho s_u^*).
\]

In order to get \( v'(w_0 - s_v^*) = Ev' (\hat{w}_1 + \rho s_u^*) \), savings must increase, i.e. \( s_v^* > s_u^* \). From continuity, this also holds if \( \beta \rho \) is sufficiently close to 1.

(ii) If \( v(c) \) is more risk averse than \( u(c) \), \( \frac{u(c)}{w(c)} \) is decreasing. Since, in the case \( \beta \rho < 1 \), (5) gives \( \hat{w}_1 < w_0 - s_u^* \), we have

\[
\frac{v'(w_0 - s_u^*)}{v' (\hat{w}_1)} < \frac{u'(w_0 - s_u^*)}{u' (\hat{w}_1)} = \beta \rho.
\]

Combining (6) and (7), i.e. higher prudence and higher risk aversion of \( v(c) \) yields

\[
v'(w_0 - s_u^*) < \beta \rho Ev' (\hat{w}_1 + \rho s_u^*).
\]

Again, savings must increase to yield equality, \( s_v^* > s_u^* \).

(iii) In analogy to the proof of (ii).

The condition given in case (ii) resembles that in Proposition 5 of Kimball and Weil (2009, 259), where the savings decision is analyzed for Kreps-Porteus preferences. Note, however, that the result in (ii) cannot be obtained as special case of their Proposition 5. The next section provides some intuition for the results of our Proposition 1.

3 Interpretation

Proposition 1 says that the level of savings is regularly affected by changes of prudence and changes of risk aversion. The impact of risk aversion crucially depends on the level of \( \beta \rho \). This can be explained as follows.
If $\beta \rho = 1$, marginal utility in period 0 coincides with expected marginal utility in period 1 for any utility function, such that some distributional balance between the periods is achieved: In the absence of risk, consumption is equal in both periods, while with stochastic wealth consumption in period 1 equals the precautionary wealth equivalent $\hat{w}_1$. Thus, by having $\beta \rho = 1$, changes in the distribution over time are a priori ruled out, and only changes of prudence affect the optimal savings behavior when the utility function is replaced.

If $\beta \rho < 1$, however, expected marginal utility in period 1 exceeds marginal utility in period 0, which means that the distribution across time is tilted towards the earlier period, i.e. $\hat{w}_1 < w_0 - s_u^*$. Even though higher prudence still induces higher savings via the precautionary savings motive, a higher savings level will now only be ensured if the new utility function is also more risk averse than $u(c)$. Then, the distribution over time becomes more equal, which, given $\beta \rho < 1$, implies that consumption in period 0 is reduced and hence savings are increased.

Finally, with $\beta \rho > 1$, the distribution over time is in favor of the future period. In order to generate more savings in this case, the increase in prudence has to be complemented by lower risk aversion which gives a more unequal distribution over time and thus a lower consumption for the earlier period, i.e. higher savings.

Since, in general, there is no systematic relationship between changes in prudence and changes in risk aversion (cf. Eeckhoudt and Schlesinger, 1994, and Maggi, Magnani, and Menegatti, 2006), our result is substantial. For specific classes of utility functions, as e.g. the isoelastic ones, an increase in prudence is accompanied by an increase in risk aversion. From our analysis it then follows that, in the case of $\beta \rho > 1$, there are opposing effects on the level of savings when the utility function changes within this class. This ambiguity has, in quite another framework, also been observed by Dasgupta (2008) in his comment on the welfare theoretic approach used in Stern’s influential “Review on the Economics of Climate Change” (Stern, 2006). In general, the change of the utility function may lead to a precautionary effect and a consumption smoothing effect over time that either support or work against each other.

In general, it is well possible that the partial effect from changing risk aversion dominates, so that changes in prudence are irrelevant for changes in savings, or that criteria based on
changing prudence or changing risk aversion cannot characterize changes in savings. We consider these cases in the next section.

4 Irrelevance and Impossibility

Low risks. If future wealth is certain, i.e. $\tilde{w}_1 = w_1$, only changes of risk aversion matter. Therefore, by continuity, for any given $u(c)$, $\beta$, and $\rho$ with $\beta \rho < 1$ and any utility function $v(c)$ that is more risk averse than $u(c)$, there always exists, irrespective of the prudence of $v(c)$, a random wealth distribution $\tilde{w}_1$ with $E\tilde{w}_1 = w_1$ such that $s_u^* > s_u^*$. If $\beta \rho > 1$, the analogous result hold for utility functions $v(c)$ that are less risk averse than $u(c)$. In this case, more saving is also compatible with lower prudence if future wealth is uncertain.

High Interest Rate. Concerning changes of prudence, another irrelevance result is obtained when, for given $u$, $\beta$, and $\rho$, the condition

$$w_0 - s_u^* \leq \tilde{w}_1 + \rho s_u^*$$

holds for $s_u^*$ and $\tilde{w}_1 := \min \tilde{w}_1$. Then, with optimal savings, wealth in period 1 in all states of the world is at least as high as wealth in period 0. This clearly requires $\beta \rho > 1$, and it is typically possible to generate the situation described in (8) by only decreasing $\rho$ strongly enough.\(^3\) Now, assume that $u(c)$ is replaced by any utility function $v(c)$ that is less risk averse than $u(c)$. Then, $h(c) := v'(c)/u'(c)$ is increasing in $c$, such that

$$v'(w_0 - s_u^*) = h(w_0 - s_u^*) u'(w_0 - s_u^*) = Eh (w_0 - s_u^*) u'(\tilde{w}_1 + \rho s_u^*)$$

$$< Eh (\tilde{w}_1 + \rho s_u^*) u'(\tilde{w}_1 + \rho s_u^*) = Ev'(w_0 + \rho s_u^*).$$

By the standard argument already applied in the proof of Proposition 1, it then follows that

\(^3\)Let $u'(c) > 0$ for all $c > 0$. Now assume that $\rho s_u^* < M < \infty$ for all $\rho > 0$. Then, from concavity $Eu'(\tilde{w}_1 + \rho s_u^*) > Eu' (\tilde{w}_1 + M) > 0$ for all $\rho$ such that, for any $\beta > 0$, $\lim_{\rho \rightarrow \infty} \beta \rho Eu'(\tilde{w}_1 + \rho s_u^*) = \infty$. The supposed boundedness of $\rho s_u^*$, however, implies $\lim_{\rho \rightarrow \infty} s_u^* = 0$, such that $\lim_{\rho \rightarrow \infty} u'(w_0 - s_u^*) = u'(w_0) < \infty$, which is not compatible with the first order condition (2). Thus, $\lim_{\rho \rightarrow \infty} (\tilde{w}_1 + \rho s_u^*) = \lim_{\rho \rightarrow \infty} \rho s_u^* = \infty$. This implies that there must exist a $\tilde{\rho}$ such that $\tilde{w}_1 + \rho s_u^* > w_0 > w_0 - s_u^*$ for all $\rho > \tilde{\rho}$. 
s_v^* > s_u^*$, independently of any assumption on the change in prudence. As we have started with a general utility function $u(c)$, these considerations also show that the potential irrelevance of changes in prudence for changes in savings is not a remote possibility, but rather a generic phenomenon.

**Different utility functions in both periods.** We finally consider the general case where the utility functions in both periods are different. By $u_0(c_0)$ we denote the utility function in the earlier, and by $u_1(c_1)$ that in the later period. Under otherwise unchanged assumptions, the objective function then becomes

$$u_0(w_0 - s) + \beta E u_1(\bar{w}_1 + \rho s).$$

(10)

We now show that, given $u_0(c_0)$, $\beta$, and $\rho$, it is not possible to characterize the class of period 1 utility functions $v_1(c_1)$ that induce higher savings than the original utility function $u_1(c_1)$ only by referring to their (absolute) degrees of risk aversion and prudence.

**Proposition 2.** Let $u_1(c_1)$ be replaced by some other utility function $v_1(c_1)$. Then, there always exists a utility function $\tilde{v}_1(c_1)$ which has the same degree of risk aversion and prudence as $v_1(c_1)$ everywhere, but induces a lower amount of savings than $u_1(c_1)$.

**Proof.** Define $\tilde{v}_1(c_1)$ as $\tilde{v}_1(c_1) := \gamma v_1(c_1)$ for some constant $\gamma > 0$. Thus, $\tilde{v}_1(c_1)$ clearly has the same degrees of risk aversion and prudence as $v_1(c_1)$. Now, choose $\gamma$ small enough such that

$$u'_0(w_1 - s^*_{u_0, u_1}) > \beta E \gamma v'_1(\bar{w}_1 + \rho s^*_{u_0, u_1}) = \beta E \tilde{v}'_1(\bar{w}_1 + \rho s^*),$$

(11)

where $s^*_{u_0, u_1}$ denotes optimal savings under the original combination $(u_0(c_0), u_1(c_1))$ of utility functions. Then, again by the standard argument described in the proof of Proposition 1, savings must decrease when $u_1(c_1)$ is substituted by $\tilde{v}_1(c_1)$.

So we see that, because of a level effect, it cannot be expected in the general case that changes of risk aversion and/or prudence will provide sensible results on changes of savings behavior.
5 Conclusion

This paper has confirmed that only in certain special cases, changes in the degree of prudence of utility functions have unambiguous effects on the level of optimal savings in the standard two period model. Only if the underlying combination of the interest rate and the pure discount rate approximately give rise to distributional neutrality across the two periods, it is ensured that higher prudence induces higher savings. Otherwise, distributional effects that are not grasped by changing prudence but instead by changing risk aversion as a separate determinant become relevant for the saving decision. It is a feature of the standard model of savings under uncertainty that the elasticity of marginal utility not only serves as a measure of relative risk aversion but also of the elasticity of intertemporal substitution. This simplifying assumption is crucial for our results. A generalization of the results in our paper in this direction is left for future research.

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References


